The Effect of Geometry on Survival and Extinction in a Moving-Boundary Problem Motivated by the Fisher–KPP Equation

Alex Tam Mat Simpson

School of Mathematical Sciences, Queensland University of Technology

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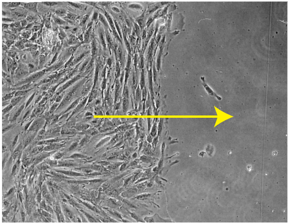
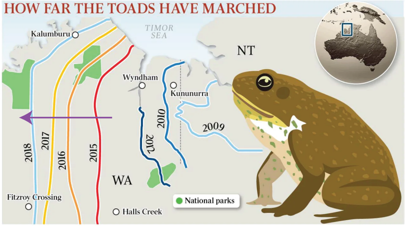
Invading and Retreating Populations

- Invading populations common in cell biology[1] and ecology.



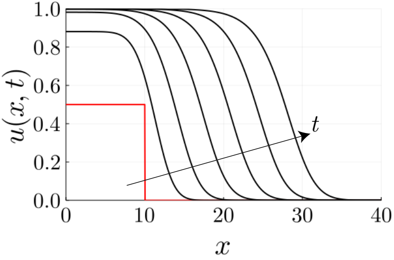


- Retreating populations can also occur.

[1] P. K. Maini, D. L. S. McElwain, and D. I. Leavesley, (2004).

Mathematical Models of Biological Invasion

- Reaction–diffusion equations often used to model populations.
  - Travelling-wave solutions capture constant invasion speed.
  - Few parameters: helps fit models to data.
- Fisher–KPP (FKPP) equation: $u_t = u_{xx} + u(1 - u)$.



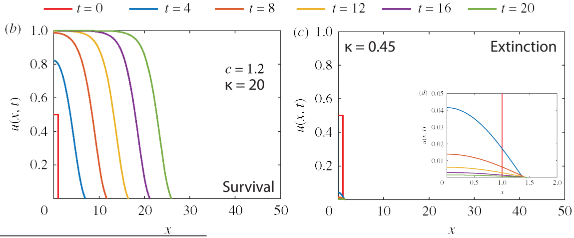
- FKPP equation has practical disadvantages.
  - Cannot identify boundary between un/occupied regions.
  - Cannot model population extinction.

Dimensionless Fisher–Stefan Model

- Recast FKPP as moving-boundary problem on $0 < x < L(t)$.

$$u_t = u_{xx} + u(1-u) \quad \text{on} \quad 0 < x < L(t),$$
$$u_x = 0 \quad \text{on} \quad x = 0,$$
$$u = 0, \quad L'(t) = -\kappa u_x \quad \text{on} \quad x = L(t),$$
$$u(x,0) = u_0(x) \quad \text{on} \quad 0 < x < L(0).$$

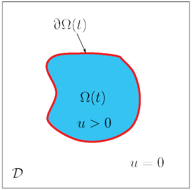
- $L(t)$ defines interface between un/occupied regions.
- Solutions with population extinction possible[2].
- Travelling wave emerges if ever $L(t) > L_{crit} = \pi/2$.



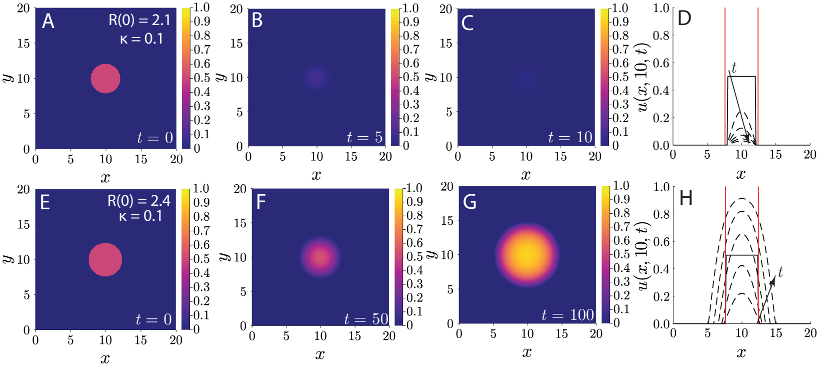
[2] M. El-Hachem, S. W. McCue, J. Wang, Y. Du, and M. J. Simpson, (2019).

2D Fisher–Stefan Model and Numerical Solutions

$$u_t = \nabla^2 u + u(1-u) \quad \text{on} \quad \Omega(t),$$
$$u = 0, \quad V = -\kappa \nabla u \cdot \hat{\boldsymbol{n}} \quad \text{on} \quad \partial\Omega(t),$$
$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \text{on} \quad \Omega(0).$$

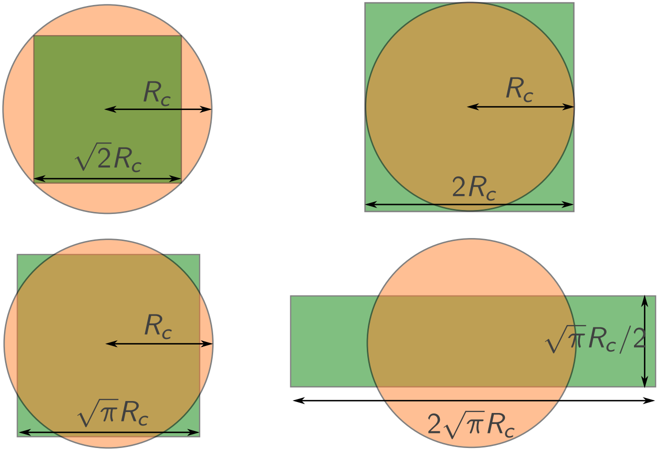


- Circular populations survive if ever $R(t) > R_c$.



Survival and Extinction in 2D

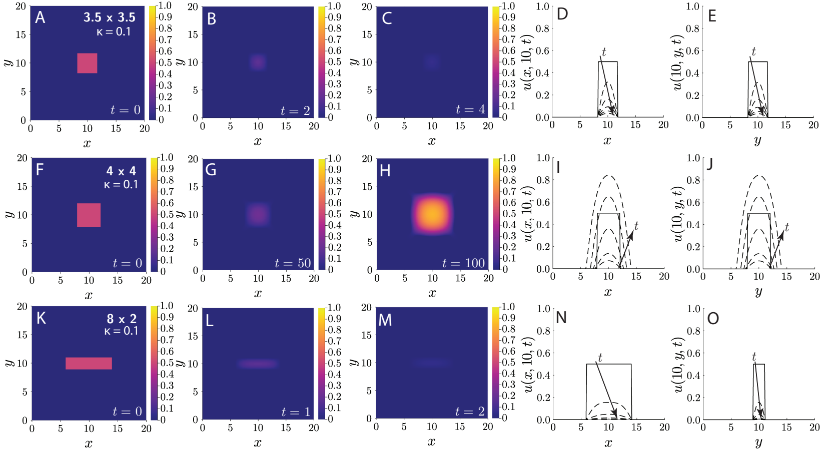
- Circular populations survive if ever $R(t) > R_c$.[3]
- Survival and extinction in general 2D geometry unexplored.



[3] M. J. Simpson, (2020).

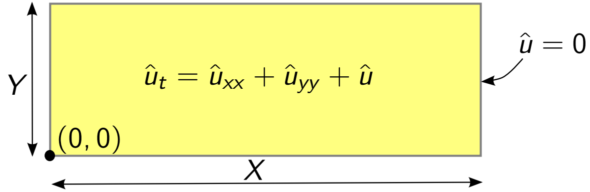
Numerical Solutions for Initially-Rectangular Regions

- Initially-rectangular populations can survive or become extinct.
- Rectangle area alone cannot explain survival/extinction.



Survival and Extinction in Rectangular Geometry

- As $u \to 0$, will population recover or become extinct?
- Leading-order solution on fixed domain.



$$\hat{u}(x,y,t) \sim A_{1,1} \sin\left(\frac{\pi x}{X}\right) \sin\left(\frac{\pi y}{Y}\right) e^{-\left(\frac{\pi^2}{X^2}+\frac{\pi^2}{Y^2}-1\right)t} \quad \text{as} \quad t \to \infty.$$

- Survival requires

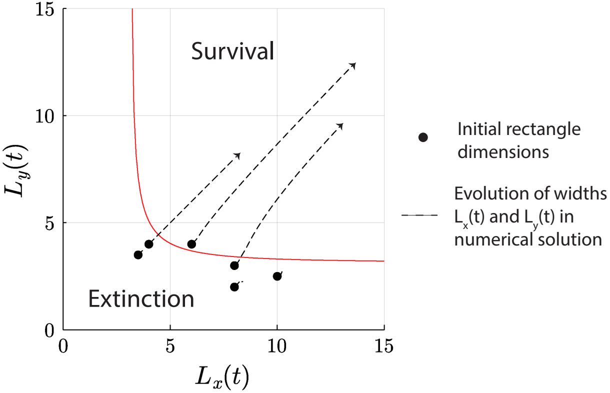
$$\underbrace{\int_\Omega \hat{u}(x,y,t)}_{\substack{\textit{Population}\\ \textit{accumulation in } \Omega}} > \underbrace{\int_{\partial\Omega} -\nabla \hat{u} \cdot \hat{\boldsymbol{n}}}_{\substack{\textit{Loss through } \partial\Omega\\ \textit{due to diffusion}}} \implies XY > \pi\sqrt{Y^2 + X^2}.$$

Rectangular Numerical Solution Summary

- Let $L_x(t)$, $L_y(t)$ be widths of $\Omega(t)$ in numerical solutions.
- Analysis suggests population survives if ever

$$L_x > \pi, \quad \text{and} \quad L_y > \pi\sqrt{\frac{L_x^2}{L_x^2 - \pi^2}}.$$

- Numerical solutions agree with analytical result.



Summary

- Fisher–Stefan model modifies Fisher–KPP equation by introducing a moving boundary.
- Knowledge of geometry necessary to predict survival and extinction in 2D populations.
- Preprint available on arXiv[4].
- Level-set software available on GitHub: alex-tam.
- Future work: Linear stability analysis of planar fronts.

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- ANZIAM 2022 Organisers

[4] A. K. Y. Tam and M. J. Simpson, (2022), submitted to *Physica D: Nonlinear Phenomena*, arXiv: 2201.06215, URL: http://arxiv.org/abs/2201.06215.