Front Stability for a Moving-Boundary Model for Biological Invasion and Recession

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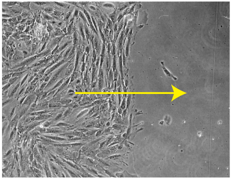
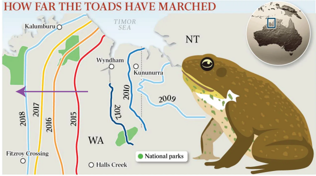
Invading and Receding Biological Populations

- Invading/receding populations common in cell biology[1] and ecology.
  - Invading: occupied region grows.
  - Receding: occupied region shrinks.



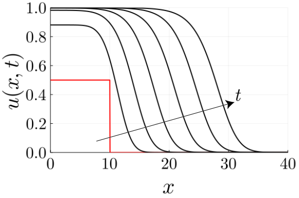


- Seek prototype models for range of phenomena:
  - Constant speed of front, i.e. travelling waves.
  - Well-defined interface between occupied and unoccupied regions.
  - Invasion and/or recession.
- Continuum, single-species population represented by density $u(\boldsymbol{x}, t)$.

[1] P. K. Maini, D. L. S. McElwain, and D. I. Leavesley, Tissue Eng. (2004).

Prototype One-Species Mathematical Models

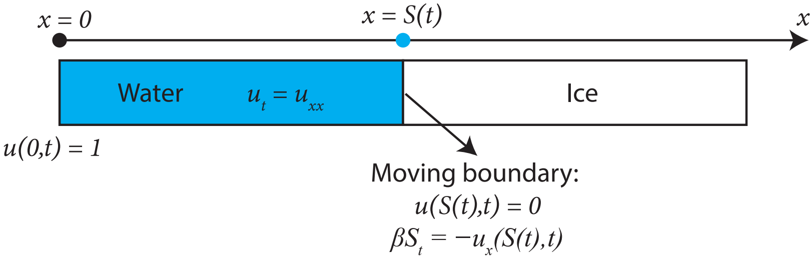
1. **Fisher–KPP** (FKPP) equation: $u_t = u_{xx} + u(1-u)$.



- Travelling waves, speed $c \geq 2$.
- Non-compact support.
- Population cannot recede.

2. One-Phase **Stefan** Problem:

- Models change of phase, e.g. ice melting, water solidification.

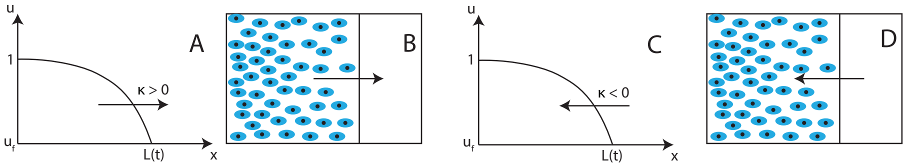


1D Fisher–Stefan Model

- Combines FKPP model with Stefan-like condition.

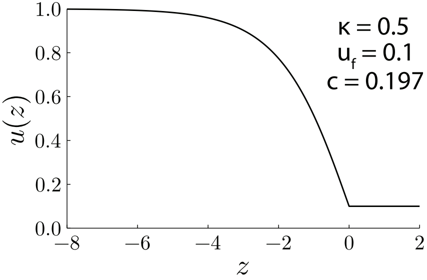
$$
\begin{aligned}
u_t &= u_{xx} + u(1-u) \quad \text{on} \quad 0 < x < L(t),\\
u(0,t) &= 1, \quad u(L(t),t) = u_{\mathrm{f}},\\
L_t &= -\kappa u_x(L(t),t), \quad L(0) = L_0\\
u(x,0) &= U(x) \quad \text{on} \quad 0 < x < L_0.
\end{aligned}
$$

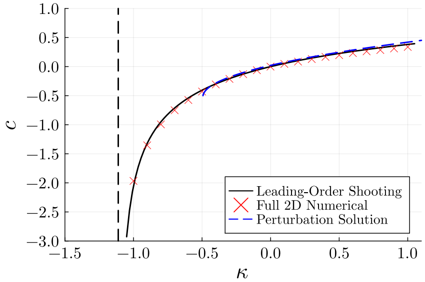
- $0 \leq u_{\mathrm{f}} < 1$, usually $u_{\mathrm{f}} = 0$ (compact support).
- Interface between occupied/unoccupied regions is the point $x = L(t)$.
- Parameter κ determines whether population invades or recedes.
    - $\kappa > 0$: invasion. $\kappa < 0$: recession.



1D Fisher–Stefan: Travelling Wave Solutions

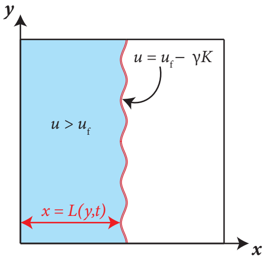
- Assume boundary moves with constant speed, $L'(t) = c$,
- Introduce travelling wave variable, $z = x - L(t) = x - L_0 - ct$.
- Biologically-relevant $(u \geq 0)$ travelling waves for $-\infty < c < \infty$.[2]
- Unique wave speed c corresponding to each $\kappa > -1/(1 - u_\mathrm{f})$.





[2] M. El-Hachem, S. W. McCue, and M. J. Simpson, Math. Med. Biol. (2022).

2D Fisher–Stefan Model



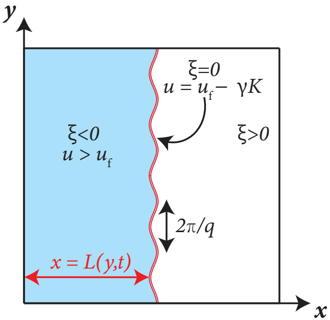
$$\begin{aligned} u_t &= u_{xx} + u_{yy} + u(1-u) && \text{on} \quad 0 < x < L(y,t), \\ u &= 1 && \text{on} \quad x = 0, \\ u &= u_f - \gamma K && \text{on} \quad x = L(y,t), \\ V &= -\kappa \nabla u \cdot \hat{\boldsymbol{n}} && \text{on} \quad x = L(y,t), \\ u(x,y,0) &= U(x,y) && \text{on} \quad 0 < x < L(y,0). \end{aligned}$$

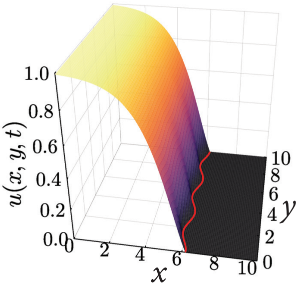
- Interface now represented by the curve $x = L(y,t)$.
- Incorporate curvature-dependent surface tension term at interface[3].
- Biology: surface tension might represent cell–cell adhesion[4].
- **Question**: Can front patterns emerge in 2D?
    - Advancing FKPP fronts are stable, but what about receding fronts?
    - How does surface tension influence results?

[3] J. Chadam and P. Ortoleva, IMA J. Appl. Math. (1983).
[4] G. Forgacs et al., Biophys. J. (1998).

Linear Stability Analysis

- Perturb front shape and population density
  - Perturbations of form $\varepsilon e^{iqy+\omega t}$: Wave number q, growth rate ω.
- Stable: $\omega < 0$. Unstable: $\omega > 0$.





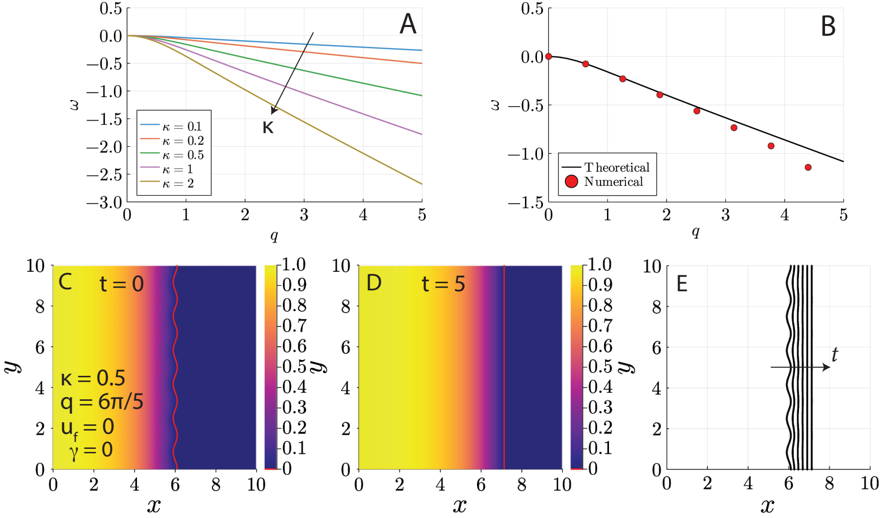
$$L(y,t) = ct + \varepsilon e^{iqy+\omega t} + \mathcal{O}(\varepsilon^2), \quad \xi = x - L(y,t) = x - ct - \varepsilon e^{iqy+\omega t}$$

$$u(\xi, y, t) = u_0(\xi) + \varepsilon u_1(\xi) e^{iqy+\omega t} + \mathcal{O}(\varepsilon^2).$$

- Leading-order solution $u_0(\xi)$ is the planar (1D) travelling wave.
- First-order correction problem determines growth rate $\omega(q)$.

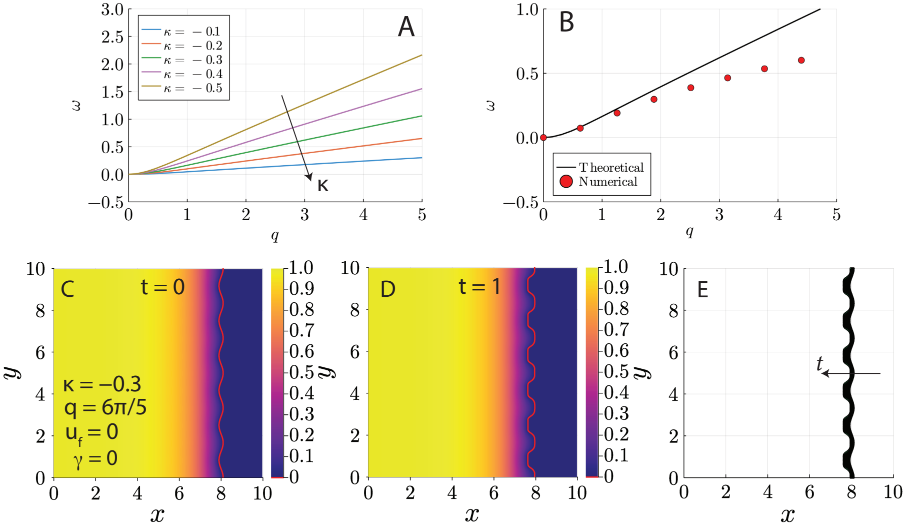
Advancing Fronts are Stable (Zero Surface Tension)

- Advancing planar waves stable to perturbations of all wave numbers.
- Consistent with FKPP equation and planar melting in Stefan problem.



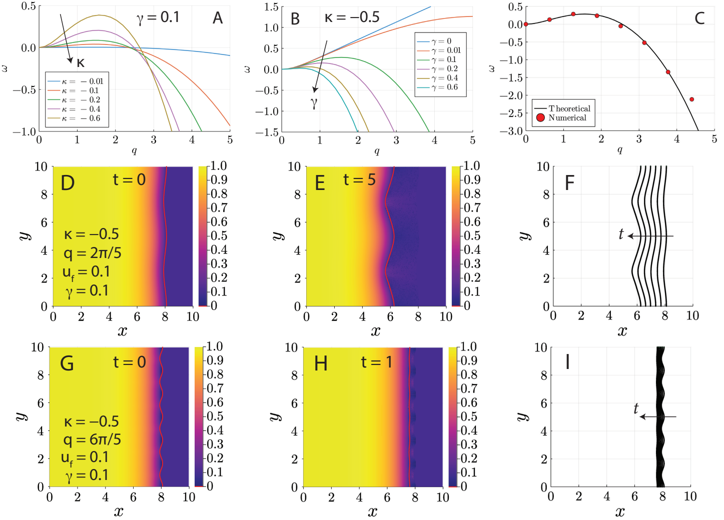
Receding Fronts are Unstable (Zero Surface Tension)

- Receding planar waves unstable to perturbations of all wave numbers.
- Consistent with planar solidification in Stefan problem.



Surface Tension Stabilises Receding Waves

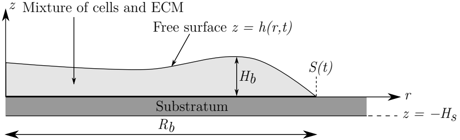
- Regularised receding fronts have instability only for small q.
- Most unstable wave number indicates preferred pattern wavelength.



Upcoming Project!

There will soon be a **2.5-year Postdoc** (based at U. Adelaide) and **PhD scholarships** (at UoA and/or UniSA) for an ARC Discovery Project.

- Understanding mechanisms that inhibit/promote biofilm expansion.
  - Agent-based modelling.
  - PDE models (reaction–diffusion, thin-film, viscous flow).
  - Scientific computing and numerics.
- Please chat if you are interested or know someone who might be!



UoA Project Team: Ben Binder, Ed Green, Jennie Gardner




- Plus international collaboration with U. Southampton and U. Kent!

Summary

- Fisher–Stefan model has travelling waves for $-\infty < c < \infty$.
- Receding fronts unstable, regularised by surface tension.
- Future work:
  - Analyse two-phase Fisher–Stefan model.
  - Stability analysis of general moving-boundary problems.

Code **Paper**

Acknowledgements:

- UniSA, Mat Simpson, ANZIAM Organisers, audience.
- I acknowledge the Kaurna and Yirrganydji people as Traditional Owners of the Adelaide and Cairns regions respectively, their Elders past and present, and their continued connection with their land, waterways and community.

Bonus: Linear Stability Boundary-Value Problem

- Both BVPs are solved on $-\infty < \xi < 0$.

Leading-Order problem for the travelling wave, $u_0(\xi)$:

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}u_0}{\mathrm{d}\xi} + u_0(1-u_0) = 0,$$

$$u_0(-\infty) = 1, \quad u_0(0) = u_\mathrm{f},$$

$$\frac{\mathrm{d}u_0(0)}{\mathrm{d}\xi} = -\frac{c}{\kappa}.$$

$\mathcal{O}(\varepsilon)$ problem for the correction, $u_1(\xi)$:

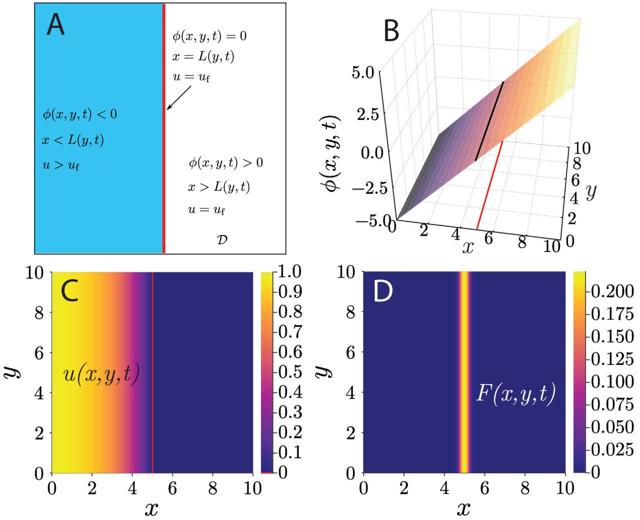
$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\xi^2} + c\frac{\mathrm{d}u_1}{\mathrm{d}\xi} + \left[1 - \omega - q^2 - 2u_0(\xi)\right] u_1(\xi) = -\left(\omega + q^2\right)\frac{\mathrm{d}u_0}{\mathrm{d}\xi},$$

$$u_1(-\infty) = 0, \quad u_1(0) = -\gamma q^2,$$

$$\frac{\mathrm{d}u_1(0)}{\mathrm{d}\xi} = -\frac{\omega}{\kappa}.$$

Bonus: Level-Set Method

- Level-set and finite-difference methods used to solve Fisher–Stefan model numerically.



Bonus: Growth Rate Saturation

- Growth rate saturates at long time in numerical solutions.