A Moving-Boundary Model for Biological Invasion and Recession in Two Dimensions

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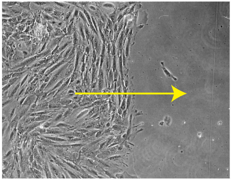
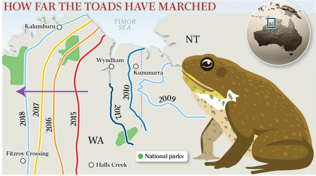
Invading and Receding Biological Populations

- Invading/receding populations common in cell biology[1] and ecology.
    - Invading: region occupied grows, population establishes.
    - Receding: region occupied shrinks, population might become extinct.



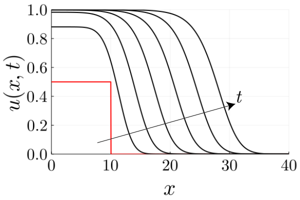


- Seek prototype models for range of phenomena:
    - Invasion and/or recession.
    - Constant speed invasion/recession.
    - Well-defined interface between occupied and unoccupied regions.
    - 1D/2D populations.
- Continuum, single-species population represented by density $u(\boldsymbol{x}, t)$.

[1] P. K. Maini, D. L. S. McElwain, and D. I. Leavesley, Tissue Eng. (2004).

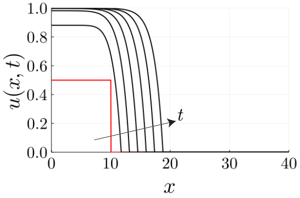
Reaction–Diffusion Models

- Reaction–diffusion equations often used for populations.
  - Travelling-wave solutions capture constant invasion speed.
  - Few parameters: helps fit models to data.
- Dimensionless Fisher–KPP (FKPP) equation: $u_t = u_{xx} + u(1-u)$.



- Travelling waves, speed $c \geq 2$
- Non-compact support
- Local density cannot decrease

- Porous-Fisher's (PF) equation: $u_t = (uu_x)_x + u(1-u)$.

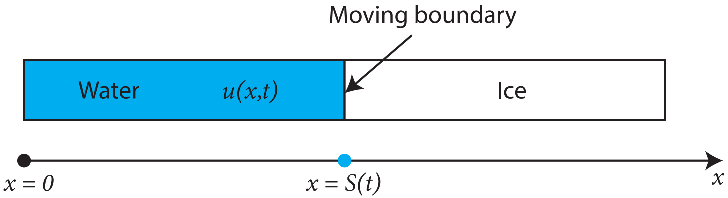


- Travelling waves, speed $c \geq 1/\sqrt{2}$
- Compact support
- Local density cannot decrease

One-Phase Stefan Problem

- PDE moving-boundary problem involving the heat/diffusion equation.
- Models change of phase, e.g. ice melting, water solidification.

$$
\begin{gathered}
u_t = u_{xx} \quad \text{on} \quad 0 < x < S(t), \\
u(0, t) = 1, \\
u(S(t), t) = 0, \\
\beta S_t = -u_x(S(t), t), \\
u(x, 0) = U(x) \quad \text{on} \quad 0 < x < S(0).
\end{gathered}
$$



1D Fisher–Stefan Model

- Solve FKPP model with Stefan-like condition.
- κ represents population loss/gain at interface.
  - $\kappa > 0$: population invades. $\kappa < 0$: population recedes.

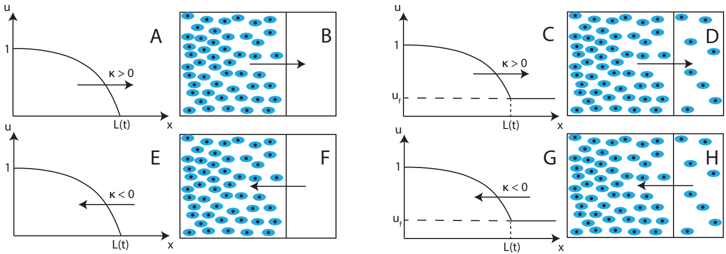
$$u_t = u_{xx} + u(1 - u) \quad \text{on} \quad 0 < x < L(t),$$

$$u_x(0, t) = 0,$$

$$u(L(t), t) = u_f,$$

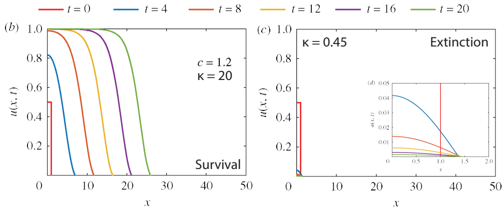
$$L_t = -\kappa u_x(L(t), t),$$

$$u(x, 0) = U(x) \quad \text{on} \quad 0 < x < L(0).$$



Travelling Waves and 1D Survival/Extinction Results

- Fisher–Stefan model first proposed by Du and Lin[2].
- Survival/extinction for 1D[3] and radially-symmetric[4] geometry.
    - Population survives if region it occupies becomes sufficiently large.
    - 1D planar: $L(t) > L_c$. Radially-symmetric: $L(t) > R_c$.
- Admits travelling wave solutions for $-\infty < c < \infty$.
    - Unlike FKPP, which only admits feasible solutions for $c \geq 2$.



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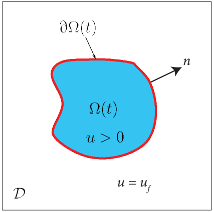
[2] Y. Du and Z. Lin, SIAM J. Math. Anal. (2010).
[3] M. El-Hachem et al., Proc. Royal Soc. A (2019).
[4] M. J. Simpson, ANZIAM J. (2020).

2D Fisher–Stefan Model and Research Questions

- Fisher–Stefan model on general 2D region $\Omega(t)$, with boundary $\partial\Omega(t)$.



$$
\begin{aligned}
u_t = u_{xx} + u_{yy} + u(1-u) \quad &\text{on} \quad \boldsymbol{x} \in \Omega(t),\\
u = u_f \quad &\text{on} \quad \boldsymbol{x} \in \partial\Omega(t),\\
V = -\kappa \nabla u \cdot \widehat{\boldsymbol{n}} \quad &\text{on} \quad \boldsymbol{x} \in \partial\Omega(t),\\
u(x, y, 0) = U(x, y) \quad &\text{on} \quad \boldsymbol{x} \in \Omega(0).
\end{aligned}
$$

Research Questions:

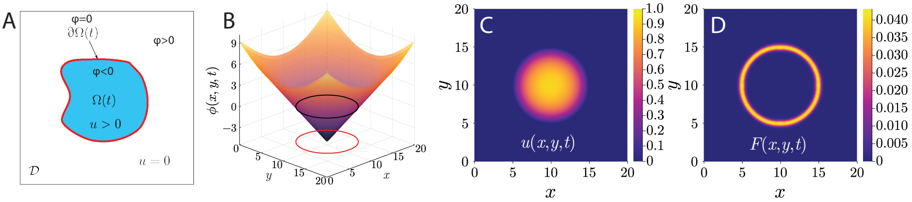
1. How does the geometry of Ω affect survival/extinction in 2D? Under what conditions will an initially-rectangular population survive?
2. Are planar fronts stable or unstable to shape perturbations? Can we predict patterns wavelength for unstable solutions?

Level-Set Method

- Embed interface as zero level-set of signed-distance function $\phi(x, y, t)$.
- Level-set method for each time step:
  1. Solve FKPP equation on $\Omega(t)$.
  2. Calculate extension velocity field: $F(x, y, t)$ such that $F = V$ on $\partial\Omega$.
  3. Evolve position of interface ($\phi = 0$) by solving level-set equation:

$$\frac{\partial \phi}{\partial t} + F\,|\nabla \phi| = 0.$$

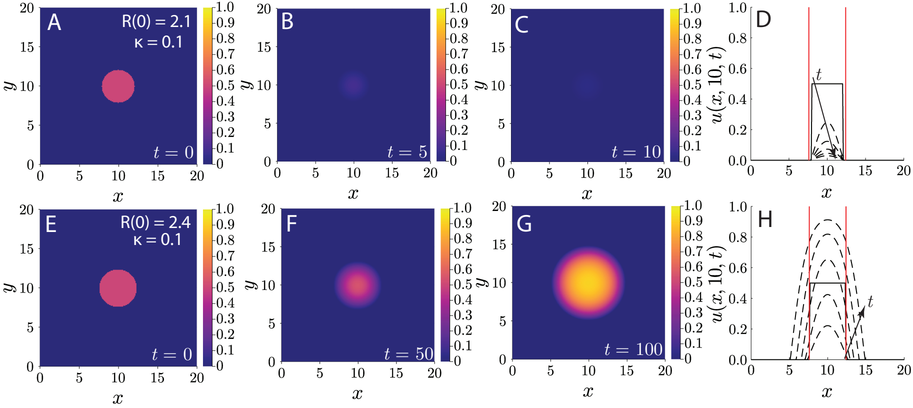
  4. Reinitialise ϕ as a signed-distance function.



- Open-source Julia code available on GitHub: alex-tam.

Survival/Extinction in Circular Geometry

- Circular populations survive if ever $R(t) > R_c$.[5]



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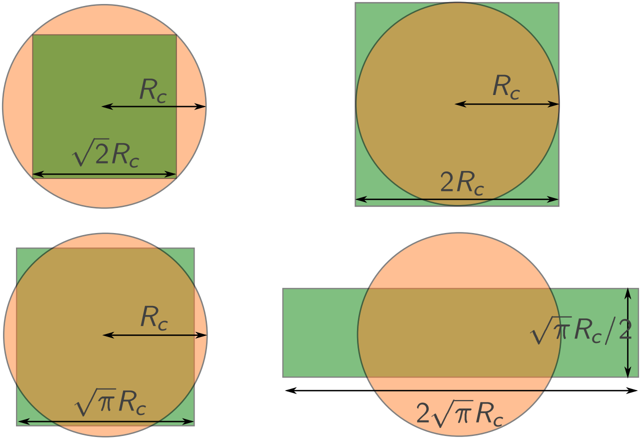
[5]M. J. Simpson, ANZIAM J. (2020).

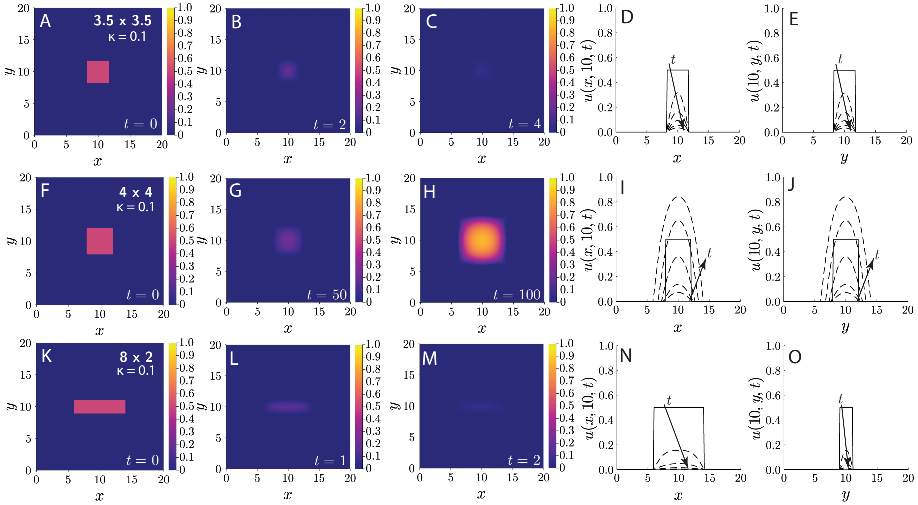
Problem 1: Survival/Extinction in 2D

- Survival/extinction in general 2D geometry unexplored.
- We consider survival/extinction in initially-rectangular regions.



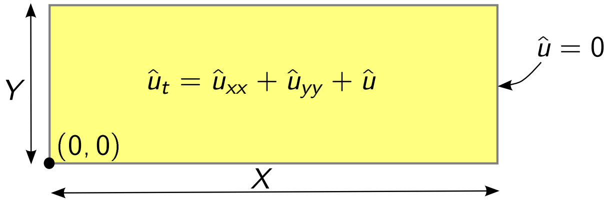
Numerical Solutions for Rectangular Regions

- Initially-rectangular populations can survive or become extinct.
- Rectangle area alone cannot explain survival/extinction.



Analytical Results: Small u

- As $u \to 0$, will population recover or become extinct?
- Consider leading-order solution on fixed domain.



$$\hat{u}(x, y, t) \sim A_{1,1} \sin\left(\frac{\pi x}{X}\right) \sin\left(\frac{\pi y}{Y}\right) e^{-\left(\frac{\pi^2}{X^2} + \frac{\pi^2}{Y^2} - 1\right)t} \quad \text{as} \quad t \to \infty.$$

- Survival requires

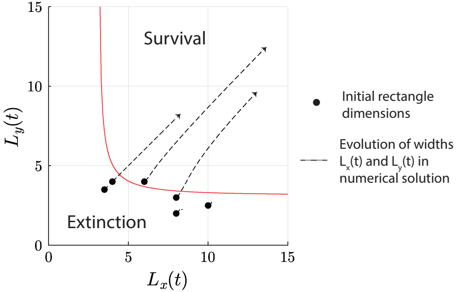
$$\underbrace{\int_\Omega \hat{u}(x, y, t)}_{\substack{\textit{Population} \\ \textit{accumulation in } \Omega}} > \underbrace{\int_{\partial\Omega} -\nabla \hat{u} \cdot \hat{\boldsymbol{n}}}_{\substack{\textit{Loss through } \partial\Omega \\ \textit{due to diffusion}}} \implies XY > \pi\sqrt{Y^2 + X^2}.$$

Summary: Survival/Extinction in 2D

- Let $L_x(t)$, $L_y(t)$ be widths of $\Omega(t)$ in numerical solutions.
- Analysis suggests population survives if ever

$$L_x > \pi, \quad \text{and} \quad L_y > \pi\sqrt{\frac{L_x^2}{L_x^2 - \pi^2}}.$$

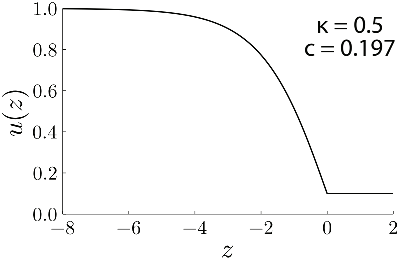
- Numerical solutions agree with analytical result.

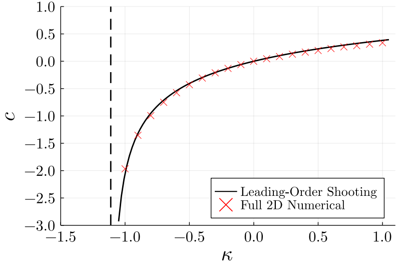


Travelling Wave Solutions

- 1D: Introduce travelling-wave co-ordinate $z = x - L(t) = x - ct$.
- Solution to boundary-value problem determines travelling wave profile.
  - Wavespeed c chosen to satisfy derivative BC at $z = 0$.

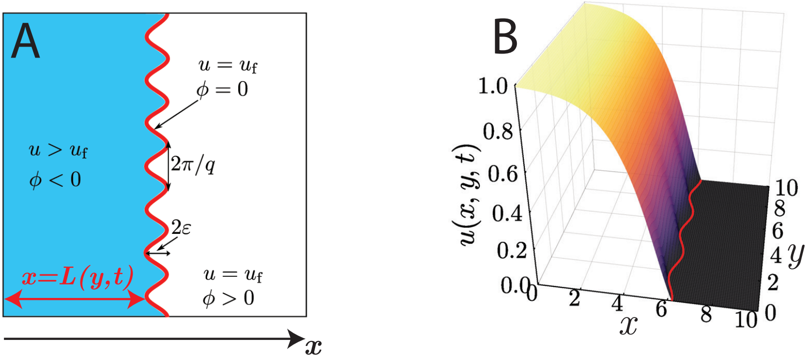
$$\frac{d^2u}{dz^2} + c\frac{du}{dz} + u(1-u) = 0 \quad \text{on} \quad -\infty < z < 0,$$
$$u(-\infty) = 1, \quad u(0) = u_f,$$
$$\frac{du(0)}{dz} = -\frac{c}{\kappa}.$$





Problem 2: Front Stability for Planar Travelling Waves

- For a travelling wave, $\Omega(t) : 0 < x < L(y, t)$.
- Periodic BC on top and bottom in numerical solutions.
- Apply sinusoidal shape perturbations to $L(t)$ and $u(z)$.
  - Perturbations of form $\varepsilon e^{iqy+\omega t}$: Wave number q, growth rate ω.
- Stable: $\omega < 0$. Unstable: $\omega > 0$.



Linear Stability Analysis

- Perturb front shape and population density.

$$L(y,t) = ct + \varepsilon e^{iqy+\omega t} + \mathcal{O}(\varepsilon^2),$$

$$\xi = x - L(y,t) = x - ct - \varepsilon e^{iqy+\omega t}$$

$$u(\xi,y,t) = u_0(\xi) + \varepsilon u_1(\xi) e^{iqy+\omega t} + \mathcal{O}(\varepsilon^2).$$

- Leading-order solution for $u_0(\xi)$ is planar travelling wave.
- First-order correction problem determines growth rate $\omega(q)$.

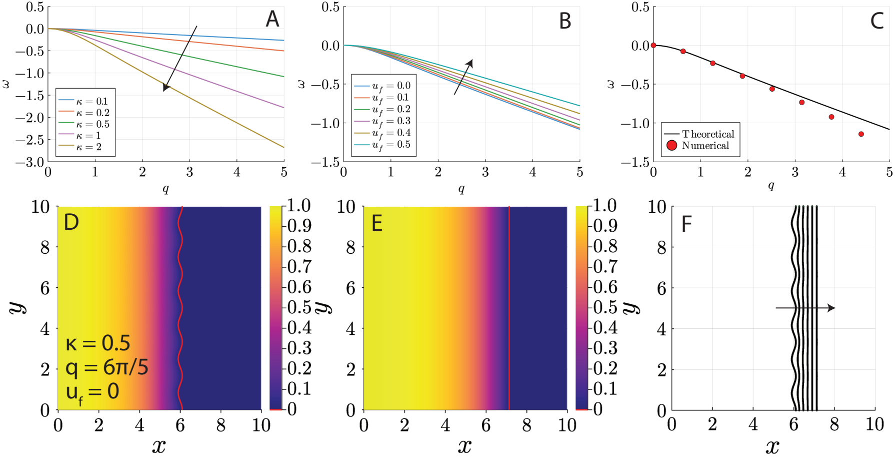
$$\frac{d^2 u_1}{d\xi^2} + c\frac{du_1}{d\xi} + \left[1 - \omega - q^2 - 2u_0(\xi)\right] u_1(\xi) = -\left(\omega + q^2\right)\frac{du_0}{d\xi},$$

$$u_1(-\infty) = 0, \quad u_1(0) = 0,$$

$$\frac{du_1(0)}{d\xi} = -\frac{\omega}{\kappa}.$$

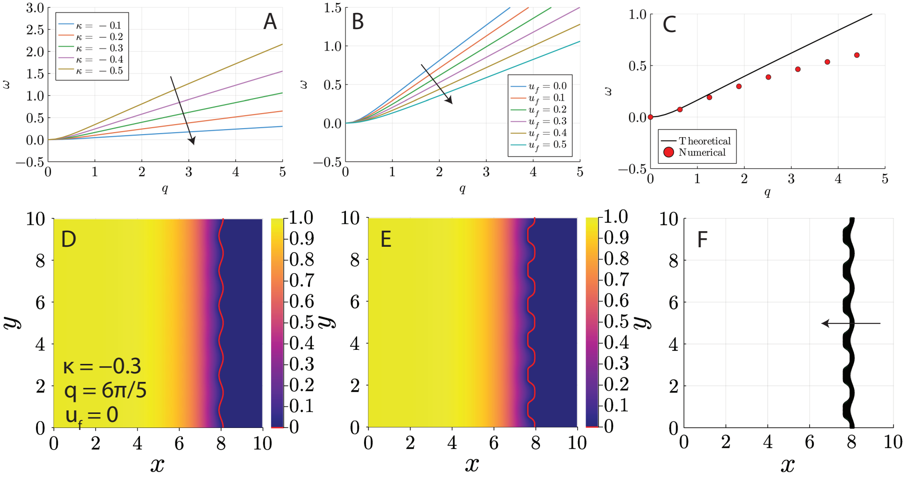
Linear Stability Results: Advancing Waves

- Advancing planar waves stable to perturbations of all wave numbers.
- Consistent with FKPP equation and planar melting in Stefan problem.



Linear Stability Results: Receding Waves

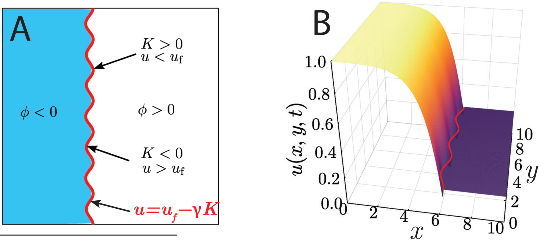
- Receding planar waves unstable to perturbations of all wave numbers.
- Consistent with planar solidification in Stefan problem.



Surface Tension Regularisation

- Modify interface condition to incorporate surface tension[6].
- Surface tension might represent cell–cell adhesion[7].

$$
\begin{aligned}
u_t = u_{xx} + u_{yy} + u(1-u) \quad & \text{on} \quad 0 < x < L(y,t), \\
u = 1 \quad & \text{on} \quad x = 0, \\
u = u_f - \gamma K \quad & \text{on} \quad x = L(y,t), \\
V = -\kappa \nabla u \cdot \hat{\boldsymbol{n}} \quad & \text{on} \quad x = L(y,t), \\
u(x,y,0) = U(x,y) \quad & \text{on} \quad 0 < x < L(y,0).
\end{aligned}
$$



[6] J. Chadam and P. Ortoleva, IMA J. Appl. Math. (1983).
[7] G. Forgacs et al., Biophys. J. (1998).

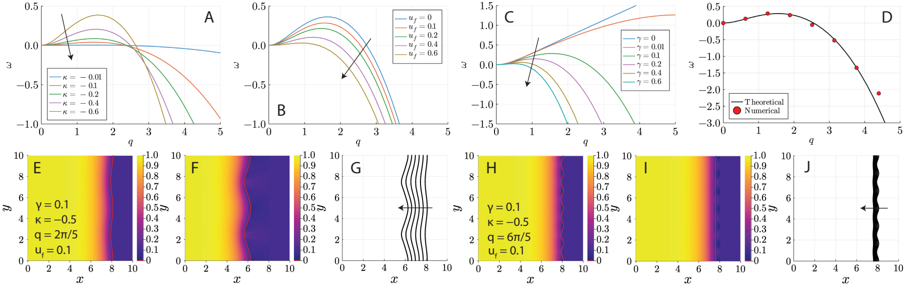
Linear Stability Results: Regularised Receding Waves

$$\frac{d^2 u_1}{d\xi^2} + c\frac{d u_1}{d\xi} + \left[1 - \omega - q^2 - 2u_0(\xi)\right] u_1(\xi) = -\left(\omega + q^2\right)\frac{d u_0}{d\xi},$$

$$u_1(-\infty) = 0, \quad u_1(0) = -\gamma q^2,$$

$$\frac{d u_1(0)}{d\xi} = -\frac{\omega}{\kappa}.$$

- Surface tension stabilises some previously unstable receding waves.
- Most unstable wave number indicates preferred pattern wavelength.



Summary

- Fisher–Stefan model involves solving FKPP equation on a moving boundary with Stefan condition.
- We considered 2 problems in 2D:
  1. Survival/extinction in initially-rectangular populations.
  2. Planar front stability and pattern formation.
- Aspect ratio influences survival/extinction for rectangular populations[8].
- Receding planar fronts are unstable, and can generate patterns[9].
- Open-source level-set method code in Julia on GitHub: alex-tam.
- Future work: Two-population model.

Acknowledgements:

- Mat Simpson.
- Seminar Organisers: Matthew Adams, Mat Simpson, Sarie Gould.
- QUT colleagues.

[8] A. K. Y. Tam and M. J. Simpson, Physica D (2022).
[9] A. K. Y. Tam and M. J. Simpson, arXiv (2022).